POLISHNESS OF THE WIJSMAN TOPOLOGY REVISITED

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Abstract. Let X be a completely metrizable space. Then the space of nonempty closed subsets of X endowed with the Wijsman topology is α-favorable in the strong Choquet game. As a consequence, a short proof of the Beer-Costantini Theorem on Polishness of the Wijsman topology is given.

Denote by $CL(X)$ the nonempty closed subsets of the metric space (X, d). The *Wijsman topology* τ_W on $CL(X)$ is the weak topology generated by $\{d(x, \cdot) : x \in X\}$, where the distance functional $d(x, A) = \inf\{d(x, a) : a \in A\}$ is viewed as a function of the set argument $A \in CL(X)$. It was shown by G. Beer in [Be1] (see also [Be2]) that given a separable complete metric space (X, d), the corresponding hyperspace $(CL(X), \tau_W)$ is Polish. Since even uniformly equivalent metrics on X may give rise to different Wijsman topologies (cf. [LL],[CLZ]), it required a separate argument to show that Polish base spaces always generate Polish Wijsman topologies. This was established by C. Costantini in [Co1]. Thus, combining these results and the fact that X embeds in $(CL(X), \tau_W)$ as a closed subspace, one gets

Theorem 1. *The space $(CL(X), \tau_W)$ is Polish if and only if (X, d) is Polish.*

It is the purpose of this note to present a short proof of the above theorem based on the so-called *strong Choquet game* (cf. [Ch] or [Ke]). In the game, denoted by Γ, two players α and β take turns in choosing objects in the topological space X with an open base $\mathcal{B}$: β starts by picking (x_0, V_0) from $\mathcal{E}(X, \mathcal{B}) = \{(x, V) \in X \times \mathcal{B} : x \in V\}$ and α responds by $U_0 \in \mathcal{B}$ with $x_0 \in U_0 \subset V_0$. The next choice of β is some couple $(x_1, V_1) \in \mathcal{E}(X, \mathcal{B})$ with $V_1 \subset U_0$ and again α picks U_1 with $x_1 \in U_1 \subset V_1$ etc. Player α wins the run $(x_0, V_0), U_0, \dots, (x_n, V_n), U_n, \dots$ provided $\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$, otherwise β wins. A *winning tactic* (abbr. w.t.) for α (cf. [Ch]) is a function $\sigma : \mathcal{E}(X, \mathcal{B}) \to \mathcal{B}$ such that α wins every run of Γ compatible with σ, i.e. such that $U_n = \sigma(x_n, V_n)$ for all n. The game Γ is α-favorable if α possesses a winning tactic; in this case X is called a *strong Choquet space* (see [Ke]). The Choquet Theorem (see [Ch], Theorem 8.7 or [Ke], Theorem 8.17) claims that a metrizable space is completely metrizable if and only if it is a strong Choquet space.

In the sequel ω will stand for the nonnegative integers, $B(x, \varepsilon)$ for the closed ball about $x \in X$ of radius ε in the metric space (X, d) and B^c for the complement of $B \subset X$, respectively. For $U \subset X$ put $U^- = \{A \in CL(X) : A \cap U \neq \emptyset\}$. It is a

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routine to show that a base $\mathcal{B}_W$ for the Wijsman topology consists of the sets of the form

$$(V_0,\dots,V_k)_D = \bigcap_{i\le k} V_i^- \cap \bigcap_{j\le m} \{A\in CL(X):\ d(x_j,A)>\varepsilon_j\}$$

with $V_i\subset X$ open $(i\le k)$, $x_j\in X$, $\varepsilon_j>0$ $(j\le m)$, $D=(x_0,\dots,x_m;\varepsilon_0,\dots,\varepsilon_m)$. For every D of this kind, denote $M(D)=\bigcup_{j\le m} B(x_j,\varepsilon_j)$.

Since metrizability of the Wijsman topology is equivalent to separability of the base space X (see [Be2], Theorem 2.1.5), in order to prove Theorem 1 it suffices by the Choquet Theorem to prove

Theorem 2. *If X is completely metrizable, then $(CL(X),\tau_W)$ is a strong Choquet space.*

Proof. Assume that (X,τ) is completely metrizable and d is a compatible metric on X. Then by the Choquet Theorem we can find a w.t. $\sigma:\mathcal{E}(X,\tau)\to\tau$ for α. Define a tactic $\sigma_W:\mathcal{E}(CL(X),\mathcal{B}_W)\to\mathcal{B}_W$ for α as follows: first, for each $V\in\tau$ and $A\in V^-$ fix a point $x_{A,V}\in A\cap V$. Then given $(A,\mathbf{V})\in\mathcal{E}(CL(X),\mathcal{B}_W)$ with $\mathbf{V}=(V_0,\dots,V_k)_D$ and $D=(x_0,\dots,x_m;\varepsilon_0,\dots,\varepsilon_m)$ define

$$\sigma_W(A,\mathbf{V}) = (\sigma(x_{A,V_0\cap M(D)^c}, V_0\cap M(D)^c),\dots,\sigma(x_{A,V_k\cap M(D)^c}, V_k\cap M(D)^c))_{\tilde{D}}$$

where $\tilde{D}=(x_0,\dots,x_m;\tilde{\varepsilon}_0,\dots,\tilde{\varepsilon}_m)$ with $\tilde{\varepsilon}_j=\frac{\varepsilon_j+d(x_j,A)}{2}$ for all $j\le m$. Then $A\in\sigma_W(A,\mathbf{V})\subset\mathbf{V}$. We will show that σ_W is a winning tactic for α.

Indeed, suppose that $(A_0,\mathbf{V}_0),\mathbf{U}_0,\dots,(A_n,\mathbf{V}_n),\mathbf{U}_n,\dots$ is a run of Γ in $CL(X)$ such that $\mathbf{U}_n=\sigma_W(A_n,\mathbf{V}_n)$ for all n. Denote $\mathbf{U}_n=(U_0^n,\dots,U_{l_n}^n)_{B_n}$ and $\mathbf{V}_n=(V_0^n,\dots,V_{k_n}^n)_{D_n}$ for appropriate B_n and D_n. Observe that $\mathbf{V}_{n+1}\subset\mathbf{U}_n$ and $\mathbf{V}_{n+1}\neq\emptyset$ implies that $M(D_{n+1})^c\subset M(B_n)^c$ and for all $s\le l_n$ there exists $t\le l_{n+1}$ such that $V_t^{n+1}\cap M(D_{n+1})^c\subset U_s^n\cap M(B_n)^c$. Hence, without loss of generality, assume that $k_{n+1}>l_n=k_n$ and $t=s$. Put $l_{-1}=-1$. Then for all $n\in\omega$ and $l_{n-1}<i\le l_n$,

$$(x_{A_n,V_i^n\cap M(D_n)^c}, V_i^n\cap M(D_n)^c), U_i^n\cap M(B_n)^c,\dots,$$
$$(x_{A_{n+j},V_i^{n+j}\cap M(D_{n+j})^c}, V_i^{n+j}\cap M(D_{n+j})^c), U_i^{n+j}\cap M(B_{n+j})^c,\dots$$

is a run of Γ in X compatible with σ, so there exists $a_i\in\bigcap_{j\in\omega} U_i^{n+j}\cap M(B_{n+j})^c$. Denote by A the closure of $\{a_i:\ i\in\omega\}$ in X. Fix n and i. Then for some $N>n$, $a_i\in U_i^N\cap M(B_N)^c\subset M(B_{n+1})^c=M(\tilde{D}_n)^c$.

Consequently, if $D_n=(x_0,\dots,x_m;\varepsilon_0,\dots,\varepsilon_m)$ then for all $j\le m$, $d(x_j,a_i)\ge\tilde{\varepsilon}_j$, thus $d(x_j,A)=\inf\{d(x_j,a_i):\ i\in\omega\}\ge\tilde{\varepsilon}_j>\varepsilon_j$. It follows now that $A\in\mathbf{V}_n$ for all $n\in\omega$, whence α wins the run. ☐

Remark. It is known that metrizable spaces that are not β-favorable in Γ are the hereditarily Baire spaces (see [De]), i.e. spaces, every *closed* subspace of which is a Baire space. In general however even strong Choquet spaces may be non-hereditarily Baire ([De]). It could be of interest therefore to find out if complete metrizability of X implies hereditary Baireness of $(CL(X),\tau_W)$. Note that it will be certainly a Baire space ([Zs]) and it could be non-Čech-complete, as was shown in [Co2].

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